Conservation laws for the nonlinear Schrödinger equation in Miwa variables.

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Abstract

A compact expression for the generating function of the constants of motion for the nonlinear Schrödinger equation is derived using the functional representation of the AKNS hierarchy.

1 Introduction.

In the present note we would like to discuss once more the conservation laws for the nonlinear Schrödinger equation (NLSE),

$$i\frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + 2q^2 r = 0 \tag{1}$$

$$-i\frac{\partial r}{\partial t} + \frac{\partial^2 r}{\partial x^2} + 2qr^2 = 0 \tag{2}$$

Existence of an infinite number of conserved quantities is an characteristic feature of integrable partial differential equations (PDE), and this question in context of the NLSE was discussed in the very first works devoted to this model, where authors established integrability of the NLSE and elaborated the corresponding inverse scattering transform (IST). The IST is based on a representation of the equation in question as a compatibility condition for a overdetermined linear system (the so-called zero curvature representation) which in the case of the NLSE can be written as

$$\frac{\partial}{\partial x}\Psi(x,t;\lambda) = U(x,t;\lambda)\Psi(x,t;\lambda) \tag{3}$$

$$\frac{\partial}{\partial t}\Psi(x,t;\lambda) = V(x,t;\lambda)\Psi(x,t;\lambda) \tag{4}$$

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Here Ψ is a 2-column, the matrix U is given by

$$U = i\begin{pmatrix} \lambda & r \\ q & -\lambda \end{pmatrix} \tag{5}$$

and V is some 2×2 matrix which is a second order polynomial in λ (in what follows we will not need its explicit form).

The U-V representation (3), (4) of the NLSE is enough to obtain answers to a wide range of questions related to this equation. In particular, one can derive from (3), (4) an infinite number of integrals of motion of the NLSE. This can be done as follows (here we will only outline some key moments, more elaborated description of the IST one can find in various textbooks on this topic, as, e.g., [1, 2, 3]). Introducing the so-called Jost functions $\Psi_\pm$ of the scattering problem (3) (i.e. solutions of (3) satisfying different boundary conditions) and the scattering matrix $T(x,t;\lambda)$, $\Psi_+ = \Psi_- T$, one can obtain from the zero-curvature representation that the diagonal elements of the 2×2 matrix T do not depend on time. Hence they (or their logarithms) can be used as generating functions of constants of motion (the later are the coefficients of the power series in λ (λ^{-1}) of the former).

Another approach stems from the viewpoint when one considers an integrable equation (the NLSE in our case) as a member of an integrable hierarchy (the AKNS in our case). The U-V representation of the equations of the AKNS hierarchy can be written as

$$\begin{aligned} \frac{\partial}{\partial x}\Psi &= U\Psi & (6) \\ \frac{\partial}{\partial t_n}\Psi &= V_n\Psi & (7) \end{aligned}$$

where U is given again by (5) and V_n's are different matrices (V_n is some nth order polynomial in λ). All equations of the hierarchy are compatible and one can solve them simultaneously, i.e. one can think of q and r as functions of an infinite set of variables $q, r = q, r(t_1, t_2, t_3, ...)$ with the evolution with respect to t_k being given by the kth NLSE (kth member of the AKNS hierarchy). In some situations such standpoint leads to more transparent results, and the main aim of this note is to apply it to the question of the description of the constants of motion of the NLSE (read constants of motion of the AKNS hierarchy).

2 Functional representation of the AKNS hierarchy.

Our starting point is the so-called functional representation of the AKNS hierarchy which can be written as

$$\begin{aligned} i\zeta\partial_1 q &= q - q^- + \zeta^2 q^2 r^- & (8) \\ -i\zeta\partial_1 r &= r - r^+ + \zeta^2 q^+ r^2 & (9) \end{aligned}$$

Here q and r are functions of an infinite set of times, $q = q(\mathrm{t})$, $r = r(\mathrm{t})$,

$$f(\mathrm{t}) = f(t_1, t_2, t_3, ...) \tag{10}$$

$\partial_1 = \partial/\partial t_1$ and the designation $f^\pm$ stands for the function with shifted arguments (Miwa shifts),

$$
\begin{aligned}
f^{\pm} &= f(\mathrm{t} \pm i[\zeta]) && (11)\\
&= f(t_1 \pm i\zeta, t_2 \pm i\zeta^2/2, t_3 \pm i\zeta^3/3, ...) && (12)
\end{aligned}
$$

This equation, which can be termed as 'AKNS hierarchy in Miwa variables', may be derived in different ways. First this can be done by careful analysis of the linear problems (6), (7). One can find the shifts $t_k \to t_k \pm i\zeta^k/k$ in some textbooks on the IST (see, e.g., chapters 3,4 of the book [2]). Another way is to use the (generalized) Hirota's bilinear identities which are one of the most important formulae of the Kyoto school approach to the integrable systems. Explicitly equations (8) and (9) have been written down in the paper [4]. One should also mention paper [5] where similar representation has been derived for the Davey-Stewartson system, from which one can easily obtain one for the AKNS hierarchy. We will not repeat here derivation of (8) and (9) and only demonstrate that the first equations of the AKNS hierarchy (the NLSE in particular) can be easily obtained from them. Indeed, using the multidimensional Taylor series for $f^{\pm} = f(\mathrm{t} \pm i[\zeta])$,

$$
f^{\pm} = f \pm i\zeta\partial_1 f + \frac{\zeta^2}{2}(\pm i\partial_2 f - \partial_{11} f) + \frac{\zeta^3}{6}(\pm 2i\partial_3 f - 3\partial_{21} f \mp i\partial_{111} f) + ... \qquad (13)
$$

(here ∂_k stands for $\partial/\partial t_k$, ∂_{jk} for $\partial^2/\partial t_j\partial t_k$, etc) and expanding equations (8), (9) in power series in ζ one will obtain that the functions q and r satisfy an infinite number of PDEs. The first non-trivial equations (the ζ^2 terms),

$$
\begin{aligned}
i\partial_2 q + \partial_{11} q + 2q^2 r &= 0 && (14)\\
-i\partial_2 r + \partial_{11} r + 2qr^2 &= 0 && (15)
\end{aligned}
$$

are nothing else than the NLSE. Hereafter we will identify variables x, t and t_1, t_2

$$
x = t_1, \qquad t = t_2 \qquad (16)
$$

The next equations

$$
\begin{aligned}
2\partial_3 q - 3i\partial_{21} q - \partial_{111} q - 6q^2\partial_1 r &= 0 && (17)\\
2\partial_3 r + 3i\partial_{21} r - \partial_{111} r - 6r^2\partial_1 q &= 0 && (18)
\end{aligned}
$$

can be rewritten using (14), (15) as

$$
\begin{aligned}
\partial_3 q + \partial_{111} q + 6qr\partial_1 q &= 0 && (19)\\
\partial_3 r + \partial_{111} r + 6qr\partial_1 r &= 0 && (20)
\end{aligned}
$$

These are the third-order NLSE. Thus one can view (8), (9) as a 'condensed' form of the AKNS hierarchy.

The key moment is that, if we deal not only with solutions of the NLSE, $q, r(x,t)$ but consider them as solutions of all equations of the hierarchy, $q = q(t_1, t_2, ...)$, $r = r(t_1, t_2, ...)$, then we can formally solve the auxiliary linear problem. Indeed, it follows from (8), (9) that the 2×2 matrix

$$\Psi = \begin{pmatrix} 1 & -\zeta r^- \\ \zeta q^+ & 1 \end{pmatrix} \begin{pmatrix} \exp(iu_1) & 0 \\ 0 & \exp(-iu_2) \end{pmatrix} \tag{21}$$

where

$$u_1 = \frac{x}{2\zeta} + \zeta \int \mathrm{d}x \; q^+ r \tag{22}$$

$$u_2 = \frac{x}{2\zeta} + \zeta \int \mathrm{d}x \; q r^- \tag{23}$$

solves (6) with $\lambda = (2\zeta)^{-1}$. Hence we can now rewrite the results which were presented in terms of solutions of (6) (i.e. in terms of the Jost or Baker-Akhiezer functions) in terms of q, r themselves. This is also valid for the generating function of the integrals of motion.

Of course, matrix (21) is a *formal* solution of (6) and one must be ready to face some problems when, e.g., one will try to construct the Jost functions (i.e. to satisfy some boundary conditions). However, for our purposes this is not an obstacle. Moreover, we will not repeat the 'classical' algorithm, Jost functions $\Psi_\pm(\lambda) \to$ scattering matrix $T(\lambda) \to$ generating function $\ln T_{11}(\lambda)$. Knowing the answer, we will first present the final result, and then, using the functional representation (8), (9) of the AKNS hierarchy, will prove it.

3 Conservation laws.

The main result of this work can be presented as follows: the function

$$J(\mathrm{t}, \zeta) = q(\mathrm{t} + i[\zeta]) r(\mathrm{t}) \tag{24}$$

is the generating function for the constants of motion.

Indeed, it follows from (14), (15) that

$$i \frac{\partial}{\partial t} q^+ r = \frac{\partial}{\partial x} \left(q^+ \frac{\partial r}{\partial x} - \frac{\partial q^+}{\partial x} r \right) - 2 q^+ r \left(q^+ r^+ - q r \right) \tag{25}$$

Using again (14), (15), this time with shifted arguments, one can easily get

$$i\zeta \frac{\partial}{\partial x} q^+ r = q^+ r^+ - q r \tag{26}$$

which leads to

$$\frac{\partial}{\partial t} J(\zeta) = \frac{\partial}{\partial x} F(\zeta) \tag{27}$$

where

$$F = i \frac{\partial q^+}{\partial x} r - q^+ \frac{\partial r}{\partial x} - \zeta \left(q^+ r \right)^2 \tag{28}$$

(recall that $x = t_1$ and $t = t_2$).

Thus we have obtained an infinite number of divergent-like conservation laws

$$\frac{\partial}{\partial t} J_m = \frac{\partial}{\partial x} F_m \tag{29}$$

where J_m's and F_m's are coefficients of the Taylor series for $J(\zeta)$ and $F(\zeta)$

$$J(\zeta) = \sum_{m=0}^{\infty} J_m \zeta^m \tag{30}$$

$$F(\zeta) = \sum_{m=0}^{\infty} F_m \zeta^m \tag{31}$$

Some first of the conserved densities are given by

$$J_0 = qr \tag{32}$$

$$J_1 = i\frac{\partial q}{\partial x} r \tag{33}$$

$$J_2 = \frac{1}{2}\left(i\partial_2 q - \partial_{11} q\right) r = -\left(\frac{\partial^2 q}{\partial x^2} + q^2 r\right) r \tag{34}$$

$$J_3 = \left(\frac{i}{3}\partial_3 q - \frac{1}{2}\partial_{21} q - \frac{i}{6}\partial_{111} q\right) r = -i\left(\frac{\partial^3 q}{\partial x^3} + 4qr\frac{\partial q}{\partial x} + q^2\frac{\partial r}{\partial x}\right) r \tag{35}$$

Note that to present J_m for $m = 2, ...$ in a standard way, i.e. in terms of q, r and their derivatives with respect to x, one has to use evolution equations of the hierarchy (19), (20) and higher. However, it is possible not to use these equations but instead to 'iterate' the identity (8), which can be rewritten as

$$q^+ = q + i\zeta q_x^+ - \zeta^2 \left(q^+\right)^2 r \tag{36}$$

$$= q + i\zeta q_x - \zeta^2 \left[q_{xx}^+ + \left(q^+\right)^2 r\right] - i\zeta^3 \left[\left(q^+\right)^2 r\right]_x \tag{37}$$

$$= \ldots \tag{38}$$

or to return to the traditional inverse scattering scheme: it follows from (36) that $J(\mathrm{t}, \zeta)$ satisfies

$$J = qr + i\zeta r\frac{\partial}{\partial x}\frac{J}{r} - \zeta^2 J^2 \tag{39}$$

which leads to the recurrence relation

$$J_0 = qr, \quad J_1 = q_x r \tag{40}$$

$$J_{m+1} = r\frac{\partial}{\partial x}\frac{J_m}{r} - \sum_{l=0}^{m-1} J_l J_{m-1-l}, \qquad m \geq 1 \tag{41}$$

One can easily identify these equations with the standard for the inverse scattering approach equations for the generating function. Thus, the main result (24) of this paper can be interpreted as follows. Equation (39), if considered as an ordinal differential equation, is the famous Riccati equation which cannot be solved explicitly for arbitrary functions q and r. However in our case q and r are not arbitrary but related by an infinite number of PDEs of the hierarchy, and it turns out that in this situation, though these restrictions do not

determine q and r uniquely, equation (39) can be solved formally and this solution is given by (24).

Equations (29) can be rewritten as

$$\frac{\partial}{\partial t} I_m = 0 \tag{42}$$

where I_m's are integrals of the densities J_m. In the case when q and r vanish (sufficiently rapidly) as $x \to \pm\infty$ I_m's are given by

$$I_m = \int_{-\infty}^{\infty} \mathrm{d}x \; J_m \tag{43}$$

In the periodical case, $q, r(x + L) = q, r(x)$,

$$I_m = \int_{0}^{L} \mathrm{d}x \; J_m \tag{44}$$

while in the case of non-trivial boundary conditions, say the finite-density ones, the integral in the right-hand side should be in some way regularized.

4 Conclusion.

To conclude we would like to note the following. The aim of this note is twofold. First we want to demonstrate that the Kyoto school approach can be used not only to reveal and study some mathematical structures behind integrable equations, but also to solve some 'practical' problems, as one discussed above. Another purpose of this paper is to attract attention to the functional representation of the AKNS hierarchy. This system is one of the first studied integrable models for which the IST has been developed, and since 70's till now the IST is the main tool to study the NLSE and related problems. As to the methods which were developed later for such equations as, e.g., KP and 2D Toda equations, for our knowledge, their application to the AKNS hierarchy is rather limited. At the same time the formulation of a problem in terms of the functional equations seems be perspective for, e.g., developing some perturbation schemes for various NLSE-related problems.

References

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